A Comment on "The Critical Transmitting Range for Connectivity in Sparse Wireless Ad Hoc Networks"

Paolo Santi* Douglas M. Blough† Henning Bostelmann‡

In [4], Santi and Blough presented a number of results concerning the asymptotic connectivity of wireless ad hoc networks. This correction includes fixes to several of that paper's results.

We recall the main parameters from [4]: n refers to the number of nodes in the network, r refers to the transmitting range of the nodes, which is assumed to be identical on all nodes, and l refers to the length of the region containing the network for the 1-dimensional case or the length of each side of the region for the 2-dimensional and 3-dimensional cases. Furthermore, $CONN_l$ is defined as the event that a randomly chosen network is connected.[1]

Theorem 4*. Assume that n nodes, each with transmitting range r, are distributed uniformly and independently at random over $R = [0, l]$. Further, assume that $rn = kl \ln l$ for some constant $k > 0$, where $r = r(l) \ll l$ and $n = n(l) \gg 1$. If $k \geq 2$ and $r(l) \gg 1$, or if $k = 2$ and $r(l) \in \Omega(1)$, then $\lim_{l \to \infty} P(CONN_l) = 1$.

Proof. With the condition stated in Theorem 4*, the original proof of Theorem 4 in [4] holds. □

When comparing Theorem 4* with the original Theorem 4 from [4], we note that the condition $r(l) \in \Omega(1)$ has been added when $k > 2$. Thus, the result does *not* hold for the special case where $r \to 0$ as $l \to \infty$. This special case is somewhat of an unnatural one but it is not ruled out by the theoretical model. If $r \to 0$ while $rn = kl \ln l$, then clearly $n \to \infty$ much faster than l and this implies that the node density is extremely high. Since we are primarily concerned with sparse scenarios, this special case does not fall within the main area of interest for the original paper, but nevertheless the additional constraint is required to make the theorem technically correct. We also note that Theorem 4* gives the precise sufficient conditions for connectivity with all necessary constraints. Thus, the sufficient condition portion of Theorem 3 in [4], which summarized results from an earlier conference

*Istituto di Informatica e Telematica del CNR, Pisa, Italy. e-mail: `paolo.santi@iit.cnr.it`.
†School of ECE, Georgia Inst. of Technology, Atlanta, GA - USA. e-mail: `doug.blough@ece.gatech.edu`.
‡Department of Mathematics, University of Florida, Gainesville, FL - USA. e-mail:`bostelm@math.ufl.edu`

[1] For full details of the probability space under which this event is defined, see [4].

paper without constraints, should be viewed as simply an order of magnitude result rather than a precise set of conditions for connectivity.

Since Theorem 8 of [4] is a generalization of Theorem 4 to higher-dimensional regions and uses the same proof technique, the same fix is required for Theorem 8 as well.

A more serious problem is that *Theorem 6 of [4] is false.* This theorem was originally given in [3] and was shown to be false in [1]. Theorem 6 of [4] is weaker than Theorem 5 of [4] (which still holds), but it covers a wider class of functions of r. Therefore, we adapt Theorem 5 of [4] to cover a broader range of functions.

Theorem 5*. Assume that n nodes, each with transmitting range r, are distributed uniformly and independently at random in $R = [0, l]$, and assume that $rn = (1 - \epsilon)l \ln l$ for some $0 < \epsilon < 1$. If $r = r(l) \in O(l^\epsilon)$, then the communication graph is not connected w.h.p.

Proof. Suppose the deployment area $R = [0, l]$ is divided into $C = \frac{l}{r}$ cells of equal length r, and let us consider three segments of R of equal length $l/3$. We call these segments the right, the center, and the left segments of R. (For simplicity, we will assume that $l/3r$ is an integer, and that n is a multiple of 3; a corresponding argument for interpolating values of l and n can easily be found.) The outline of the proof is as follows: First, we prove that there is a constant, non-zero probability of having at least one empty cell in the center segment of R. Then, we show that there is at least a constant, non-zero probability of at least one node appearing in the left segment and in the right segment of R. Given these facts, we prove that the event "there is at least one node in the left segment, there is at least one empty cell in the center segment, and there is at least one node in the right segment", which corresponds to a node deployment generating a disconnected communication graph, occurs with constant, non-zero probability. This implies that, under the hypotheses of the theorem, the communication graph is not connected w.h.p.

Let us first consider the center segment. We want to prove that the event $E =$ "at least one of the $\frac{C}{3}$ cells of length r is empty" asymptotically occurs with constant, non-zero probability. Let n' denote the number of nodes out of the n overall network nodes landing in the center segment. We have:

$$Prob(E) \geq \sum_{i=0}^{n/3} Prob(E|(n' = i)) \cdot Prob(n' = i) \ .$$

We observe that $Prob(E|(n' = i))$ is a decreasing function of i: as the number n' of nodes landing in the center segment increases, and given that the number of cells in the center segment is not influenced by n', the probability of having at least one empty cell cannot increase. Thus, we can write:

$$Prob(E|(n' = i)) \geq Prob(E|(n' = n/3)) \quad \text{for each } i = 0, \ldots, n/3 \ .$$

It follows that:

$$Prob(E) \geq Prob(E|(n' = n/3)) \cdot \sum_{i=0}^{n/3} Prob(n' = i) \ . \tag{1}$$

It is easy to see that $\sum_{i=0}^{n/3} Prob(n' = i) = Prob(n' \leq n/3) \geq 1/2$. As for the other term in the right hand side of (1), we observe that we have reduced our problem to an occupancy problem with $n' = n/3$ balls thrown uniformly and independently at random into $C' = C/3$ cells. We recall that we are considering a situation in which $rn = (1-\epsilon)l \ln l$ and $r = O(l^\epsilon)$. With these hypotheses, the following results[2] hold from occupancy theory [2, Ch. I, §1, Theorem 1]:

$$E[\mu] \sim C' l^{\epsilon-1}, \qquad Var[\mu] \sim C' l^{\epsilon-1}, \tag{2}$$

where μ is the random variable denoting the number of empty cells. Our goal is to show that under the condition $n' = n/3$,

$$\exists q < 1 \; \forall l > 0 : \; Prob(\mu = 0) < q. \tag{3}$$

To that end, assume that (3) is false; i.e. we can find some sequence (l_j) on which $Prob(\mu = 0) \to 1$. We know by hypothesis that $r \in O(l^\epsilon)$, which means that $C' l^{\epsilon-1} = l^\epsilon/3r$ is bounded from below by a positive constant. By passing to a suitable subsequence of (l_j), we can achieve that either $l^\epsilon/3r \to \lambda$ for some $\lambda > 0$, or that $l^\epsilon/3r \to \infty$. We will handle these two cases separately.

(i) $l^\epsilon/3r \to \infty$, and thus $E[\mu] \sim C' l^{\epsilon-1} \to \infty$. In the terminology of [2], this means that we are in the right-hand intermediate domain, and it is known that the limit distribution of μ is Gaussian. In view of Eq. (2), we conclude that $Prob(\mu = 0) \to 0$ on the sequence in question, in contradiction to our assumption.

(ii) $l^\epsilon/3r \to \lambda > 0$. In the terms of [2], we are then in the right-hand domain, and the limit distribution of μ is the Poisson distribution with parameter λ. This results in $Prob(\mu = 0) \to e^{-\lambda} < 1$, leading us to a contradiction.

This shows that Eq. (3) holds in all cases. It follows immediately that

$$Prob(E|(n' = n/3)) = 1 - Prob((\mu = 0)|(n' = n/3)) > 1 - q.$$

Let us now consider the following event E_1:"both the left and the right segment contain at least one node", and let $n'' = n - n'$ be the number of nodes in the left or right segment of R. We can write:

$$\begin{aligned} Prob(E_1) \quad &\geq \sum_{i=\frac{2n}{3}}^{n} Prob(E_1|(n'' = i)) \cdot Prob(n'' = i) = \\ &= \sum_{i=0}^{n/3} Prob(E_1|(n' = i)) \cdot Prob(n' = i) \; . \end{aligned}$$

We observe that $Prob(E_1|(n' = i))$ is a decreasing function of i: as i increases, the number n'' of nodes landing in the left or right segment decreases and, given that the number of cells in these segments is not influenced by n'', the probability of having both segments containing at least one node cannot increase. Thus, we can write

$$Prob(E_1|(n' = i)) \geq Prob(E|(n' = n/3)) \quad \text{for each } i = 0, \ldots, n/3 \; ,$$

[2]For two functions $f(x)$ and $g(x)$, we use the notation $f \sim g$ to indicate that $f(x)/g(x) \to 1$ as $x \to \infty$.

and it is clear that for large l, the following estimate holds:

$$Prob(E_1|(n' = n/3)) \geq 1/2 \ .$$

To prove the theorem, it is sufficient to observe that, conditioned on a certain value of n' (and, consequently, of $n'' = n - n'$), the two events E and E_1 are independent, i.e. $Prob((E \wedge E_1)|(n' = i)) = Prob(E|(n' = i)) \cdot Prob(E_1|(n' = i))$. Given this fact, we can write:

$$\begin{aligned}
Prob(E \wedge E_1) &\geq \sum_{i=0}^{n/3} Prob((E \wedge E_1)|(n' = i)) \cdot Prob(n' = i) = \\
&= \sum_{i=0}^{n/3} Prob(E|(n' = i)) \cdot Prob(E_1|(n' = i)) \cdot Prob(n' = i) \geq \qquad (4) \\
&\geq Prob(E|(n' = n/3)) \cdot Prob(E_1|(n' = n/3)) \cdot \sum_{i=0}^{n/3} Prob(n' = i) \geq \qquad (5) \\
&\geq Prob(E|(n' = n/3)) \cdot Prob(E_1|(n' = n/3)) \cdot Prob(n' \leq n/3) \geq \\
&\geq \frac{1-q}{4}
\end{aligned}$$

for large l, which concludes the proof of the theorem. □

In [4], Theorems 4, 5, and 6 are combined in Theorem 7. Thus, given the above analysis, we now have a modified version of Theorem 7 also.

Theorem 7*. Assume that n nodes, each with transmitting range r, are distributed uniformly and independently at random over $R = [0, l]$ and assume that $rn = kl \ln l$ for some constant $k > 0$. Further, assume that $r = r(l) \ll l$ and $n = n(l) \gg 1$. If $k = 2$ and $r(l) \gg 1$, or if $k > 2$ and $r(l) \in \Omega(1)$, then the communication graph is connected w.h.p. If $k \leq (1 - \epsilon)$ and $r = r(l) \in O(l^\epsilon)$ for some $0 < \epsilon < 1$, then the communication graph is not connected w.h.p.

Note that the simulation results reported in [4] to validate Theorem 7 are still valid and can be used to validate Theorem 7*, as they refer to the case $k = 2$ and $r = r(l) \gg 1$, for which the statement of Theorem 7 are reported in [4] is correct.

References

[1] H. Bostelmann, "Statistical Analysis of Quality Measures for Mobile Ad Hoc Networks," M.Sc. Thesis, FernUniversität Hagen, 2005. Available as e-print cs.NI/0504004.

[2] V.F. Kolchin, B.A. Sevast'yanov, V.P. Chistyakov, *Random Allocations*, V.H. Winston and Sons, Washington D.C., 1978.

[3] P. Santi and D. Blough, "An Evaluation of Connectivity in Mobile Wireless Ad Hoc Networks," *Proceedings of the International Conference on Dependable Systems and Networks (DSN '02), pp. 89–98, 2002.*

[4] P. Santi and D. Blough, "The Critical Transmitting Range for Connectivity in Sparse Wireless Ad Hoc Networks," *IEEE Transactions on Mobile Computing,* Vol. 2, No. 1, pp. 25–39, Jan.-Mar. 2003.