On Benacerraf's identification problem from the point of view of mathematicians

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1 Prologue

A 1965 essay by Paul Joseph Salomon Benacerraf (1930 – 2025) [1] begins by telling the story of two child-mathematicians, Ernie and Johnny, each strictly educated by their respective parent-mathematicians in a particular version of Peano arithmetic [2] having a single model/representation in set theory. However, the models are different in the two families. We follow the separate evolution of the two child-mathematicians until they meet and by exchanging their notes they realize that there are "theorems" (note the quotation marks) that are verified in one model but not in the other, and they start arguing.

Furthermore, it is assumed that a Platonist philosopher has followed the entire cultural evolution of the two families and is now reasoning about which could be the "true" model of Peano arithmetic among the two considered. A Platonist philosopher by definition should assume that only one of the two models is the "true" one (i.e. the model that embodies the essence of the number in all respects) or perhaps neither of the two, but certainly not both. The Platonist philosopher reasons in the manner of Aristotelian logic, that is, according to the law of excluded middle (tertium non datur) and avoiding contradictions (a proposition cannot be simultaneously true and false), furthermore for reasons unknown to us he is prone to apply the principle of insufficient reason [3]. So the author, abandoning the two boys to their arguments in the background, observes the reasoning of the Platonist philosopher.

Broadly speaking, the initial step consists in tracing the phenomenon of the "true/false theorems" in the two models back to an elementary characteristic of the two models. Namely, the fact that the same natural number $n \geq 2$ is identified (using the symbol '=') with two different sets, let's call them $I(n)$ and $I'(n)$, in the two models[1]. But the transitive property of equality would lead us to state that $I(n) = I'(n)$ when instead they are evidently different sets[2].

Not wanting to give up the transitive property of equality, the Platonist philosopher is now reduced to the situation in which he must declare only one of the two models, or neither, "true" (i.e. the model in which the equality between number and set is the authentic one) and the other false, or both, (i.e. a model for which the equality between number and set is fictitious, superficial, perhaps useful for making calculations, but essentially devoid of profound meaning). In order not to end up like Buridan's donkey, always undecided between the two equivalent piles of hay, applying the principle of insufficient reason, he concludes that both models must necessarily be false.

Here enters the author, who has observed benevolently the life, thoughts and beliefs of the two child-mathematicians and their families until now, and has also observed perhaps with less benevolence the Platonist philosopher who, following the thread of his own reasoning, has certified autonomously and without any external help the failure of the enterprise of identifying the essence of the number with that of the set. The author generalizes by observing that there are infinite possible models of Peano arithmetic that can be constructed in set theory, but for every pair of different models one falls into the same trap that the Platonist philosopher has constructed for himself. From here on the author becomes the protagonist forgetting about the Platonist philosopher, the two families of mathematicians, and the strange phenomenon of the "theorems" verified in one model but not in another.

Note the author's basic indulgence for the two families of mathematicians, and instead the satisfaction in seeing the rival Platonist philosopher fall into his own net and thus abandon the field.

What happens next? We are not interested now in establishing who is right or wrong between the Platonist philosopher and the author on the essence of the number. What we narrate from here on cannot (should not) be used either for or against the two philosophical schools that confront each other in the essay. Instead, we are interested in the story of the two families of mathematicians beyond the narrow scope of the essay. As in 'Carnage' (Roman Polanski's 2011 film) the two families

[1] For the numbers 0 and 1, however, the corresponding sets in the two models coincide.

[2] For example, the number 3 is associated in one model with the set $\{\phi, \{\phi\}, \{\phi, \{\phi\}\}\}$, while in the second model with $\{\{\{\phi\}\}\}$, where ϕ indicates the empty set.

organize a meeting (without the children) to settle their differences and reach an agreement, behind the backs and without worrying too much about the other characters in the story, and what they might say or think about them. The two families (father and mother) are the Montagues for Ernie and the Capulets for Johnny.

2 Dialogue

2.1 First day

Montagues (Model I) and Capulets (Model II) meet in front of a good mug of beer, the tension begins to dissolve and the two couples feel at ease, to the point of being able to ask the fateful question: "Could you show us the set-theoretic model that you use to develop your Peano arithmetic? Let's call it PA." At which point, each showing the other their model, both couples see a knowing smile appear on the other's face.

Capulets: "Dear Montagues, there would be no need for words between us seasoned mathematicians, but since we are here to clarify, we must declare ourselves. Please, after you. "

Montagues: "Certainly, dear Capulets, we are afraid, as we are sure you are also thinking, that we are faced with a case of double abuse of notation. That is, the symbol '=' should not be understood as a symbol of equality/identity between numbers and sets, instead it takes the place of a more precise notation that defines a function f with domain among the natural numbers N and co-domain among the sets defined by the recursive scheme:

$$f(0) \stackrel{\text{def}}{=} \phi$$

$$f(x) \stackrel{\text{def}}{=} f(x-1) \cup \{f(x-1)\} \quad se \quad x > 0$$

for model I, which we believe is due to John von Neumann [4] so we could call this family of sets I_{vn}. Similarly we will have in model II another function g with domain among the natural numbers N and co-domain among the sets defined by the recursive scheme:

$$g(0) \stackrel{\text{def}}{=} \phi$$

$$g(x) \stackrel{\text{def}}{=} \{g(x-1)\} \quad if \quad x > 0$$

This model is due, we believe, to Ernst Zermelo [5] so we will call this family of sets with I_{ez}. Interestingly, if $x \in N$ is 0 or 1 we get the same sets, then from $x \geq 2$ we get different sets."

Montagues: "It is interesting that these encodings are 1-1 and therefore invertible, that is, given a set $a \in I_{vn}$ there exists a unique number $x \in N$ such that $f(x) = a$, which allows us to define the inverse function f^{-1} with domain on I_{vn} and with co-domain in N. Similarly, we have a 1-1 function g between N and the sets in I_{ez} and its inverse function g^{-1} with domain in I_{ez} and co-domain N. Combining the functions f and the function g^{-1} we have a 1-1 function with domain in I_{ez} and co-domain in I_{vn}. Combining f^{-1} and g we have its inverse function. This allows us to uniquely and invertibly associate any set in one model with a single set in the other model without even having to mention the number to which both correspond."

Capulets: "Interesting, we could conclude that the two representations are isomorphic and furthermore both preserve the Peano axioms, so the theorems of PA derivable in the two models are the same."

Montagues: "Certainly the notation has become cumbersome, but it should save us from falling victim to misunderstandings if we were to interpret, God forbid, the symbol '=' as equivalence/identity, which can only lead to disastrous consequences."

Capulets: "We seem to recall that Dedekind demonstrated that all models for second-order Peano arithmetic are isomorphic to each other way back in 1888. Who knows if this has anything to do with it."

2.2 Second day

Montagues: "The mystery of propositions that are true in one PA model but turn out to be false in the other remains unsolved. But we do not think we can attribute this to elementary issues such as abuse of notation or problems with the transitivity of the equality/identity relation."

Capulet: "Perhaps we should consult the giants on whose shoulders we are comfortably sitting to scan the horizon. We believe that Model Theory could help us, or Goedel's theorems. Let's look into the sacred texts, such as for example the recount that I found in a 2004 course in Mathematical Logic at MIT in Boston [6]."

3 Reconciliation between mathematicians

Once it was established quite easily that the sign = does not indicate identity but is used as an abbreviation for $\stackrel{\text{def}}{=}$ which indicates the definition of a function from one domain (natural numbers) to another (sets of Zermelo-Fraenkel [7]), the two families established that it is not possible to apply the transitive properties of identity in dealing with these definitions.

The apparent anomaly of propositions that seem to be theorems if evaluated with sets I_{vn}, preferred by Ernie, while they do not seem to be theorems if evaluated with sets I_{ez}, preferred by Johnny, and vice versa, remains to be solved.

Back to basics, they consult a nice MIT course on mathematical logic from 2004 that is easily found online [6]. Lesson 17 deals with interpretations, that is, how to translate a proposition from the language of Peano arithmetic (LA) to another language, in our case the language of Zermelo–Fraenkel set theory (LS).

In the language of set theory we have the symbols of first-order logic and only one non-logic binary predicate 'belongs to' indicated by $\in$.

In the language of arithmetic we have the symbols of first-order logic and some predicates (binary and ternary) that indicate: successor (S), addition (A), multiplication (M), exponentiation (E) and 'less-than' (L). We must therefore have at hand LS formulas that replace such arithmetic predicates (S, A, M, E and L) in addition we need formulae $N(x)$ and $Z(x)$ that represent the predicates "x is a set associated with a natural number" , and "x is a set associated with zero" where x takes values among all the sets. These formulas will obviously be different for the systems ez and vn, but the translation rules that follow do not depend on which of the two systems is used, just choose one consistently.

At this point an arithmetic formula in LA language is translated into a formula in LS quite mechanically, except for one detail, that of the quantifiers. The expression $\forall x$... and $\exists x$... in the LA language indicate that x has values in the domain of numbers. The same expression in the language of sets LS indicates that x as has values in the domain of sets. However, there are sets that are not associated with numbers, so the way of translation must take this into account. In particular, one translates $\forall x P(x)$ in LA with $\forall x(N(x) \to P(x))$ in LS, where $P(x)$ is any formula in which x is a free variable. Note that for sets in LS that do not correspond to numbers the antecedent of the implication is always false so the implication is true for such sets.

Similarly, $\exists x P(x)$ in LA translates to $\exists x(N(x) \wedge P(x))$ in LS, Note that for sets in LS that do not correspond to numbers the first part of the conjunction is always false, so the conjunction is false for such sets.

Let's now see what happens to the theorem in question in LA proposed by Ernie, which states that a number x (set) is less than another number y (set) if and only if x is an element of y. Here n is the generic function that associates numbers to sets: So let's start from a formula in LA plus the function n :

$$\forall x \forall y(x < y \leftrightarrow n(x) \in n(y)) \tag{1}$$

becomes in the LS language:

$$\forall x \forall y(N(x) \to (N(y) \to ((L(x,y) \leftrightarrow x \in y))) \tag{2}$$

For Ernie who uses the von neuman system N, L and n are: N_{vn} L_{vn} and n_{vn} so the theorem established by Ernie in the LS language is:

$$\forall x \forall y(N_{vn}(x) \to (N_{vn}(y) \to (L_{vn}(x,y) \leftrightarrow x \in y))) \tag{3}$$

At this point Johnny's father and mother look at what has been written and agree that for them too this is a theorem that their son can accept (even if a little vacuous since the antecedents $N_{vn}(x)$ and $N_{vn}(y)$ are always false on the sets I_{ez}, except for the sets corresponding to 0 and 1).

The mutual recognition argument obviously also works for Johnny's theorems once they are correctly translated into the LS language, i.e. using $(S_{ez}, A_{ez}, M_{ez}, E_{ez}, L_{ez})$ and N_{ez}, Z_{ez} in the translation, with the same rules as above for treating quantifiers.

We note that no deep theorem of logic was needed to reconcile the two boys, we did not have to invoke Geodel's incompleteness theorems, or worry about whether every true formula of PA is formally provable, etc... We did not even have to invoke model theory and worry too much about axioms. Here only a little syntactical foresight resolved the situation.

At this point two of the mathematical difficulties that emerged in the discussion on page 54/55 of [1] have evaporated. Are there any others left?? There is the question of the cardinality of the set that represents a number, but we will leave this for another day, since each day has enough trouble of its own. Furthermore, the discussion on cardinalities is introduced with the aim of refuting some theories of Gottlob Frege (1848 - 1925), but in other respects it is tangential to the main discussion in [1].

We note the mathematical style in solving the question using when possible only syntactic rules and the snail trick, which brings along its little house in the shape of a shell. In this case the theorems must bring with them all their antecedents in order to be able to pass from one language to another, otherwise there is a risk of losing pieces in the move. This syntactic rigor, however, magically becomes semantic precision since it leaves no ambiguity in the translation and therefore leaves no room for false dilemmas.

4 The dialogue resumes: third day

Montagues: "It seems to us that we all agree that the asserted "true/false theorems" in the two LS models of PA are not theorems because they do not respect the rules for translating a preposition from one language to the other. On the other hand, the propositions stated in a syntactically correct way are instead equally true (therefore by definition theorems) or false in both models."

Capulet: "We wonder if we have not been too rigid in our educational approach that has taught only one model in the LS language to our young people."

Montagues: "My wife and I exclude it. In fact, the rules for translating from LA to LS are formulated to correctly treat sets that do not correspond to numbers versus sets that correspond to numbers in the model. These rules must be used regardless of whether one is aware of the existence of models other than the one adopted."

Capulet: "The time has come to bring the boys in and explain to them how theorems are born."

5 Impact on the discourse of the Platonist philosopher

On page 56 second paragraph in [1], the Platonist philosopher summarizes the situation in a bifurcation for which he must admit:
(A) that simultaneously $3 = 3_{vn}$ and $3 = 3_{ez}$, or

(B) that at least one of the two systems (*ez* or *vn*) is not correct (for various reasons), but perhaps also that neither of them is correct.
and furthermore that:
(C) there are no other alternatives (even if this is not explicitly stated).
The discourse continues, abandoning (A) very quickly because of the transitivity of equality and the fact that obviously $3_{vn} \neq 3_{ez}$. Then the text begins to deal with (B) with a long discussion that ends on page 62 with the conclusion that neither of the two systems is correct.

What to say in light of the reconciliation between Ernie's and Johnny's families. Can this reconciliation have an impact on the discourse of the Platonist philosopher? It would be easy now to show that in reality (C) does not hold because the alternative exists, that is, to give the symbol $=$ its correct meaning in the context, which is not that of identity. In other words, we have a case of the "false dichotomy" fallacy [8]. However, there is still something useful and subtle to extract from the discussion on (B). This discussion, reduced to the bare bones, is an application of the "principle of insufficient reason" in the Boolean context.

6 The principle of insufficient reason

The "principle of insufficient reason", also called the "principle of indifference", [3] is used in various interpretations of probabilistic phenomena or phenomena due to belief. In essence this principle establishes that if we do not have good reasons to attribute different probabilities to similar events, then all such events should be attributed the same probability. For example, in an ideal experiment in which a 6-sided die is thrown, each face is attributed a probability 1/6 of appearing after the throw (since for reasons of symmetry we have no reason to consider one face less probable than another). Another use of this principle in the Bayesian theory of probability as belief consists in attributing a uniform *a priori* probability to events about which we know nothing (not having yet done the appropriate experiments that can modify the *a posteriori* probability).

It is known that this principle in probability must be treated with care, since an unqualified use leads for example to Bertrand's paradoxes [9] and other similar ones. The debate is still open whether such paradoxes are inevitable or resolvable with appropriate limitations (see [10]).

The use of this principle in the Boolean context involves a translation from the continuous numerical domain of probabilities to the discrete Boolean domain of true/false values (excluding a third possibility).

A simplified version of the thread of reasoning followed by the Benacerraf Platonist is the following: there are two visions (*vn* and *ez*) for which one wants to establish whether it is true/false that they faithfully represent in all respects the natural numbers as sets. An exhaustive table of the possibilities in matrix form is the following:

$$\begin{pmatrix} & vn & ez \\ a) & true & true \\ b) & true & false \\ c) & false & true \\ d) & false & false \end{pmatrix} \tag{4}$$

By the law of the excluded middle and by the principle of non-contradiction one and only one of the 4 lines a), b) c) or d) must be correct (to the exclusion of all the others). Line a) is excluded because it corresponds to the situation A) that the Platonist philosopher believes to be contradictory, thus false. Lines b) and c) treat the conditions *vn* and *ez* differently, while, following the principle of insufficient reason, one should treat *vn* and *ez* equally by giving the two visions the same truthness/falseness value, so one can exclude b) and c). In the end only d) survives and necessarily must be the correct conclusion.

This reasoning scheme can certainly be extended to k visions, one can build a table with k columns and with 2^k rows in which the first row has all the 'true' values, the last one has all the 'false' values and in the middle the remaining $2^k - 2$ rows are mixed (with both true and false values). The principle of insufficient reason will lead to excluding all mixed lines, so that in the end one will have to choose only between the first and the last line. If there is an auxiliary argument to exclude one of the two surviving lines, the other wins.

We have already seen how in the light of the reconciliation between the mathematicians, we have an indication that this auxiliary argument on A) is not so compelling. But let's reason beyond that.

7 How to make at least one pigeon disappear

In this short paragraph I intend to show how the "principle of insufficient reason" in the Boolean/Discrete context is problematic even when applied alone (i.e. without the need for auxiliary arguments to discriminate between two surviving possibilities). To do this we will apply it in parallel to the "pigeonhole principle" [11] and observe what comes out.

Suppose we have $n+1$ ideal classical pigeons (they do not die, do not give birth, do not wander around in the dark, and cannot be in a state of quantum superposition) that go to sleep in n pigeonholes at night, for $n \geq 2$.

The pigeonhole law assures us that there will be at least one pigeonhole containing two or more pigeons.

Let us now look at the same problem from the point of view of the Benacerraf Platonist philosopher armed with the Boolean/Discrete insufficient reason principle. The Platonist asks the mathematician: "Can you say that the first hole always contains two or more pigeons? "

To which the mathematician replies that he cannot say so. The Platonist then asks the mathematician the same question about the second, the third, and so on up to the n-th hole, always receiving the same negative answer. So reasoning, the Platonist concludes that with respect to the n holes we are in the same condition of knowledge/ignorance and we have no reason to treat them differently, therefore we will treat them all equally, so each hole must have the same number of pigeons. If this number is zero, we have succeeded in the enterprise of making $n+1$ pigeons disappear, if this number is 1, then we will have concluded that a pigeon has disappeared. We can exclude that this number is two or greater than two since we would have to create ex-nihilo at least $n-1$ pigeons, something that we exclude as absurd in our ideal model. So finally having examined all the possible numbers we know that at least one pigeon has disappeared and no hole contains two or more pigeons.

8 More on pigeons: Can experimental proof help?

The pigeon holes are placed very high up and no one wants to take a ladder at night to go and look in every hole and count the pigeons it contains. Both the mathematician and the philosopher are very convinced of their reasoning and each would like the other to spend time and effort in an experimental verification. However, not even an experimental verification would be conclusive here, since real pigeons, unlike ideal ones, can die, give birth, hang around at night, and even be in quantum superposition (like Schroedinger's cats). It would probably be better to do experiments with inanimate objects such as tin cans and glass marbles. But here too, what can an experiment with 10 cans and 11 marbles tell us definitively for the case $n > 11$? Faced with these practical and theoretical difficulties, the two give up on the experimental test.

9 Irony of philosophy

In the classical myth of Plato's cave, humanity is analogized to prisoners who have been forced since childhood to live at the bottom of a cave with their backs turned to the exit and who see of the objects of the external world only the shadows that the sunlight casts on the bottom of the cave. The poor prisoners wonder about the nature of what they see, but they are naturally confused. The philosopher is the one who, having broken his chains, goes out, and with difficulty sees the objects in their true nature and returns to the cave to reveal the truth to his unfortunate companions and thus free them too [12].

In Benacerraf's article our two child-mathematicians are placed and chained with their backs to the exit in two different caves (even if at a certain point they communicate) and each of the two sees external objects illuminated by the sun (the natural numbers) only the shadow on the bottom of his cave (an interpretation of the natural numbers as sets), and furthermore they are two very different shadows. The poor child-mathematicians wonder about the nature of what they see, but they are naturally even more confused than their previous colleagues, who at least saw the same shadow to discuss. Benacerraf's Platonist philosopher, however, fails in his existential purpose because, although he has left the cave, he still reasons about shadows instead of objects in their true nature, and above all he does not return to console his companions who remain chained. On an ethical level, the Platonist philosopher conceived by Benacerraf is a very poor Platonist.

10 On the usefulness of paradoxes in philosophy and mathematics

Western philosophy and its paradoxes were born more or less together around the seventh century BC in the cultural area of classical Greece. Zeno's paradoxes of motion and quantity are among the oldest reported and still discussed today [13]. Among modern paradoxes, the most famous are perhaps Russell's set-theoretic ones [14].

The usefulness of paradoxes lies in the stimulus they provide in developing new philosophy or new mathematics or both, as well as in the intimate satisfaction of being able to undermine with little stories of a few lines understandable by everyone, abstruse theoretical constructions that occupy several volumes. They are a stimulus to keep thoughts under control when these fly too far from the ground.

Reading excerpts from Benacerraf's article widely reported in [15] and [16] I asked myself whether the dilemma described in sections I and II is actually a paradox of mathematics. I am not aware that mathematicians have set out to fix arithmetic and thus overcome this dilemma. Benacerraf's article is widely cited in the philosophical literature and is credited with having given rise to at least two new lines of inquiry in the philosophy of mathematics. It is not so obvious what could determine the fact that a non-problem for a mathematician could instead become a central problem in a branch of philosophy. So I thought it was worth exploring this borderline phenomenon between two disciplines to understand how they differ even when they are apparently talking about the same thing.

11 Zermelo, Von Neumann and Benacerraf

Ernst Friedrich Ferdinand Zermelo (1871 - 1953) [5] and John von Neumann (1903 – 1957) [4] were two mathematicians active at the beginning of the 20th century in the area of the foundations of Mathematics (and for John von Neumann in many other areas of Mathematics). Many of the main results in Mathematical Logic reported in the aforementioned MIT course [6] date back to the mid-1930s.

The article by Paul Joseph Salomon Benacerraf (1930 – 2025) is from 1965 so everything discussed by the two child-mathematicians and their parents and the way they reconciled themselves were very well known at the time this article was written. So I don't think the author was ignorant of the fact that the proposed paradox was actually empty for mathematicians.

Here comes into play the essential fact that it is not the author who observes the scene but a supposed Platonist philosopher who is obsessed with the notion of identity, and whose thoughts the author only tries to untangle, until these same thoughts turn against the initial hypothesis. So even if the author sees the Platonist getting bogged down in his own reasoning he will do nothing to bring him back on the right path, already savouring his final defeat.

In other words, accepting the mathematicians' solution from the beginning in order to avoid the apparent paradoxes implies giving up on posing the problem of what number is in terms of the identity of the concept of number with something else (here with sets in the Zermelo-Fraenkel theory). Benacerraf's Platonist instead strongly wants to cast the problem as a problem of identity (in the purest philosophical tradition) and therefore willingly accepts the misunderstanding of confusing $=$ with $\stackrel{\text{def}}{=}$ in examining Ernie and Johnny's systems (who in turn, raised in caves, cannot warn the Platonist of the misunderstanding). Nor does the author intend to stop the Platonist in his mad rush, in the end anyway he would have arrived sooner or later at the same conclusion, that "numbers cannot be sets" which was where he wanted the Platonist to arrive, but by the longest possible route. The problem of the "true/false theorems" of the two models seems more like a red herring.

Figure 1: John von Neumann

12 Obsession with definiteness

A pedestrian view of the relationship between the work of the mathematicians and that of the philosophers would see the mathematicians as intent on defining, specifying, analyzing every aspect of the problem down to the smallest comma, avoiding every ambiguity even with the help of formal languages instead of natural ones.

Furthermore, this view would see the philosophers instead at ease among the ambiguities of natural language, capable of confidently making great conceptual leaps supported by centuries of habit of logical/philosophical demonstration starting from the Platonic dialogues and Aristotelian logic up to modern Dialectics, passing through Medieval Scholasticism.

Instead, the sad story of the pigeons that disappeared because of the poorly applied principle of insufficient reason demonstrates that the mathematicians, through the pigeonhole principle, do not need to see or know exactly which hole contains two or more pigeons, it is enough for them to know that such hole exists without knowing which one it is. They can also give a symbolic

Figure 2: Ernst Zermelo

Figure 3: Paul Benacerraf

name to that hole, even without knowing which hole it is, and from there continue with their demonstrations (see the sophisticated combinatoric theory due to Frank P. Ramsey (1903 – 1930) [17]). The mathematicians does not worry about the indefiniteness of the situation, indeed they exploits it to their advantage.

On the contrary, the Platonist philosopher described in [1] in order to attain absolute and granitic knowledge by dotting all the i's and crossing all the t's is led to ask wrong/useless questions trying to identify exactly where the infamous two or more pigeons might be. At which point, after n tiring questions, having made sure of the total equivalence from the cognitive point of view of the n holes, he feels authorized to use the principle of insufficient reason to make at least one pigeon disappear in order to make the pigeons' accounts add up as if he were a magician. The excess of precision together with poorly applied principles has led him to erroneous conclusions, without his being aware of it. For the case of set-theoretic models of natural numbers, the fallacy of false dichotomy also intervenes to complicate things a little more by combining with the obsession for definiteness and the principle of insufficient reason.

13 Epilogue

Benacerraf's article is composed of three parts ("Gallia est omnis divisa in partes tres") of which the first and second are the 'pars destruens' in which the Platonist philosopher dominates who reluctantly at the end of the second part must admit that "numbers cannot be sets".

The third part begins a 'pars construens' discussion in which he tries to tackle the problem of what numbers can be, if they cannot be sets.

In this short essay we concern us with sections I and II, and we say nothing about section III. We have shown how the paradoxes invoked are easily solvable in mathematics, so they are considered non-problems and deserve at most a footnote. We have then seen that the Platonist's way of reasoning is fraught with hidden dangers due to a casual use of the principle of insufficient reason in a context foreign to that in which it originated. However, if the reasoning is problematic, it does not follow that the conclusion is false (i.e. the conclusion "numbers cannot be sets" may very well be the correct one).

The mathematical dilemmas from which the discussion in [1] starts are not such for the mathematicians themselves, who operate in a systematic way and are attentive to the syntax of the languages used (LA and LS) and to the rules on how to translate one into the other. If the Platonist philosopher in [1] had considered the solution to the apparent paradox advanced by the mathematicians then he would probably have looked for a different way, even if it is not certain that he could have found a different solution.

The mathematicians behave like the generals of an army with heavy logistics that carries slowly and with considerable effort over long distances everything it may need because one never knows what traps the enemy has devised and one must be ready to ward off any eventuality. Benacerraf's Platonist philosopher behaves like the general of a light army who bets everything on speed and travels without a pack of baggage in enemy territory, living off what the territory has to offer, trying to keep the enemy always unbalanced and unable to react.

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A Notes

1) A recent paper by Arnon and Gramayr [18] is a useful reference point to summarize the state of the art on the academic discussion and the 'fortune' of the identification problem. The

central thesis of the paper [18] is that there are valid reasons to prefer one of the two models compared (the von Neumann one) therefore an important step of the argument in the 1965 paper [1] would fall through. The issues highlighted in [18] are very different from those exposed in this essay.

2) In the MIT course cited [6], the predicate "less than" in the LS language for the von Neumann model is defined as

$$L(x, y) \stackrel{\text{def}}{=} x \in y \tag{5}$$

which would however make Ernie's theorem a tautology (independently of the antecedents). This undesired effect can be circumvented by adopting an alternative and equally valid definition of the type:

$$L(x, y) \stackrel{\text{def}}{=} \exists z(N(z) \land \neg Z(z) \land A(x, z, y)) \tag{6}$$

where we say that x is less than y if there exists a number z different from zero for which the addition of x and z is equal to y. Of course the predicate $A(x, z, y)$, as well as $N()$ and $Z()$ in LS must be defined without using $L(x, y)$ to avoid circularity in the definitions.

3) In the comparison between the pigeonhole principle and the principle of insufficient reason in Section 7 the philosopher's question about each specific hole could be changed or varied, but the result would remain the same. The problem lies in trying to infer knowledge about a specific hole knowing with certainty which hole we are talking about.

4) From the MIT course [6] it is clear that there are several variants of Peano arithmetic, which differ mainly in the type of quantifications allowed (first-order theories quantify on numbers, second-order theories quantify on numbers and sets of numbers) and in the way of treating the axioms of induction. Finally, there are other variants in which some axioms can be exchanged with some theorems, or auxiliary axioms can be associated, formally derivable from the other axioms, but which are added to the necessary axioms for convenience in the demonstrations. The discussion presented in the article [1] does not depend on these details, and for simplicity we refer to a first-order theory.