Computational Geometry

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Lecture 2: Area of a simple polygon

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Summary of Lecture 2

Reminder of of Lecture 1:

- Applications of Geometric techniques.
- Representation of polygons.
- Segment intersection, turning tests.
- Simplicity test.
- Convexity test.

Summary of lecture 2:

- Area of aTriangle,
- Area of a Polygon.

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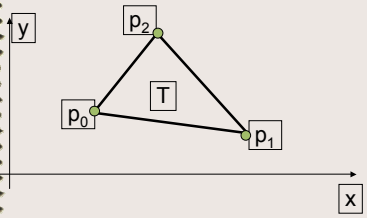
How do we compute the area of a triangle?

- We are given a triangle T with vertices in anti-clockwise order $T = (p_0,p_1,p_2)$. The coordinates of $p_i = (x_i,y_i)$ for i=0,1,2.
- How do we find the area of T?



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The high-school method.

Area(T) = (1/2) x (base) x (altitude).

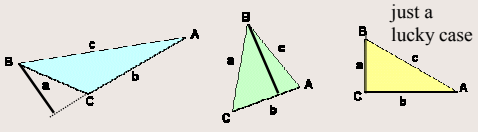
If we take p_0p_1 as base, $|p_0p_1| = \sqrt{(x_0-x_1)^2+(y_0-y_1)^2}$

But what is the altitude h?

Using a bit of trigonometry you can find out:

$$altitude(p_2) = |p_0p_2|\sin(\angle p_2p_0p_1)$$

Problem: we are using functions such as sqrt, sin, etc that are likely to have large round-off errors. This causes numerical instabilities. This happens, for example, when the angle $p_2p_0p_1$ is very small.



just a lucky case

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An alternative method

Avoiding strange functions

- What we need is an expression for Area(T) that:
  1) uses the coordinates of the vertices.
  2) uses only operations + and x.
- We use results for Linear Algebra.
- Given two vectors A and B.
  The **CROSS PRODUCT** of the vectors is a vector C such that:
  C is orthogonal to the plane of A and B.
  the direction of C is given by the skew rule.
  The module of C is the area of the parallelogram of sides A and B.

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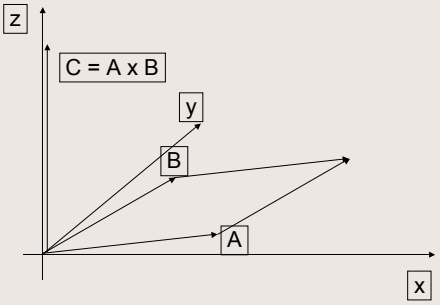
Cross Product of two vectors



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Algebraic form of the cross product.

- Vector A has components (A_x, A_y).
  Vector B has components (B_x, B_y).
- The unit vector on the x-axis is i,
  The unit vector on the y-axis is j,
  The unit vector on the z-axis in k.

$$\begin{vmatrix} i & j & k \\ A_x & A_y & 0 \\ B_x & B_y & 0 \end{vmatrix} = (A_xB_y - A_yB_x)k$$

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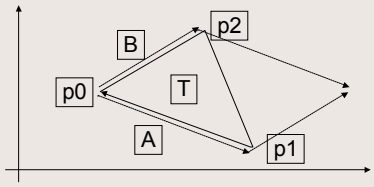
A formula for the area of a triangle.

- Taking vector A = p1-p0 and vector B = p2-p0,
  The area of T is half of the area of the parallelogram of sides A and B. So
  Area(T) = (1/2)(AxBy -AyBx) =
  (1/2)((x1-x0)(y2-y0) -(y1-y0)(x2-x0)).



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Observations

- We have reached our objective of a formula for the area of a triangle that involves directly the coordinates of the vertices and uses only sum and product.

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A mnemonic formula for the area

(if you know determinants)

- A easy way to remember Cross-product formula of the area is by noticing that the following determinant gives the value of 2Area(T).

$$\begin{vmatrix} x_0 & y_0 & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 2area(T)$$

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A second mnemonic form

(useful for manipulations)

- Let us define a function f of two points:

$$f(p_i, p_j) = x_i y_j - y_i x_j$$

- Note that $f(p_i,p_j) = - f(p_j,p_i)$.
- 2Area(T) = f(p0,p1) + f(p1,p2) + f(p2,p0), where T =(p0,p1,p2) in counterclockwise order.
- Note that (p0,p1), (p1,p2) (p2,p0) are the EDGES of T in counterclockwise order.

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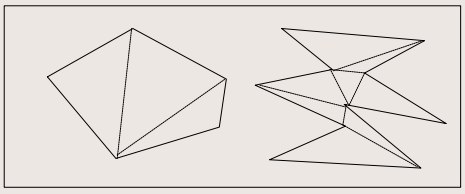
Area of Polygons

- If we are given a (convex or simple) Polygon P =(v0,..,vn) in counterclockwise order we can compute the area of P by:
  1) Decompose P into triangles,
  2) Sum the areas of these triangles.



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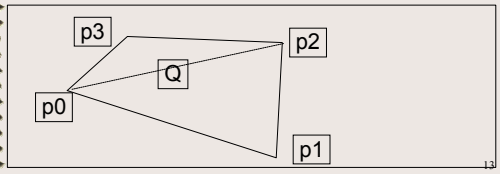
Area of polygons

- The area of a polygon does not depend on how we subdivide the polygon into triangles
- There are many ways to split a polygon into triangles.
- Is it possible to compute the area even without the first triangulation step?



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Area of quadrilateral Q

- 2Area(Q) = 2Area(p0,p1,p2) + 2Area(p0,p2,p3) =
  = f(p0,p1) + f(p1,p2) + f(p2,p0) +
  f(p2,p3) + f(p3,p0) + f(p0,p2) =
  = f(p0,p1) + f(p1,p2)+ f(p2,p3) + f(p3,p0).
- The term corresponding to the chosen diagonal (p0,p2) cancel out. The terms remaining correspond to the edges of Q in clockwise order.
- We can extend the same calculation to any convex polygon P and the terms of the diagonals cancel out and the only terms left are those of the edges in clockwise order.
- 2Area(P) = f(p0,p1) + f(p1,p2), +...,+ f(pn,p0) when P is convex and P=(p0,p1,...,pn).

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Simple, non-convex polygons

(the same formula holds)

- If P =(p0,p1,...,pn) is simple and non-convex, we can split it into triangles T1,...,Tk.
- We write the formula with the f-term of edges for each triangle.
- Each diagonal belongs to two triangles and is summed in one direction for one triangle and in the opposite direction for the other triangle.
- After all simplifications only the f-terms of the edges in counterclockwise order are left.

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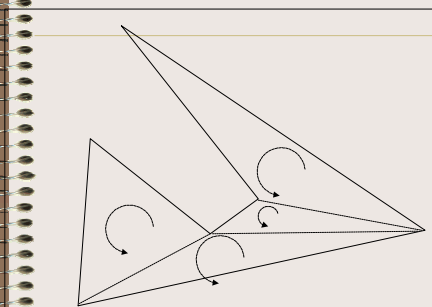
An example to show the cancellations.



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Exercise

- Implement the code for computing the area of a simple polygon.
  Hint: look at O'Rourke's example at page 26 (Code 1.3).
- Test the code on the following polygons:
- P1 = ((3,3),(10,5),(3,7))
- P2 = ((1,1)(3,3)(3,5)(2,4)(1,5))
- A simple non-convex polygon of your choice with 10 vertices.

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Conclusions

- When we design basic procedures we should avoid functions prone to numerical errors (sqrt, division, trigonometric, trascendental, etc..)
- For the area of a triangle we have a formula involving only sum and products of the coordinates of the vertices.
- An easy formula exists also for the area of simple (convex and non-convex) polygons.
- We will see more tests based on the same ideas.

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